

# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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TECHNICAL MEMORANDUM 1334

THE EFFECT OF HIGH VISCOSITY ON THE FLOW AROUND  
A CYLINDER AND AROUND A SPHERE

By F. Homann

Translation of "Der Einfluss grosser Zähigkeit bei der  
Strömung um den Zylinder und um die Kugel," ZAMM,  
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## THE EFFECT OF HIGH VISCOSITY ON THE FLOW AROUND

A CYLINDER AND AROUND A SPHERE\*<sup>1</sup>

By F. Homann

For the determination of the flow velocity one is accustomed to measure the impact pressure, i.e., the pressure intensity in front of an obstacle. In incompressible fluids the impact pressure is  $\gamma v^2/2g$  ( $\gamma$ , kg/m<sup>3</sup>, is specific weight and  $v$ , m/sec, is velocity) if the influence of viscosity can be neglected. Such an influence is appreciable, however, when the Reynolds number corresponding to impact tube radius is under about 100, and must consequently be considered, if the velocity determination is not to be faulty. The first investigations of this influence are included in the work of Miss M. Barker<sup>2</sup>. In the following pages, experiments will be reported which determine the intensity of impact pressure on cylinders and spheres; furthermore a theory of the phenomenon will be developed which is in good agreement with the measurements.

The research apparatus consists of an oil circulation in which the velocity of the oil can be varied from 0.5 centimeter per second to 30 centimeters per second with the help of a vane-type pump lying entirely in the oil. A Russian bearing oil and a mixture of this with fuel oil is used for the measurements. Figure 1 illustrates the test setup. In this is indicated: P, the pump; a, turning vanes; G, straightener; and V, the actual test section which possesses a breadth of 0.148 meter, a depth of 0.15 meter calculated from the oil surface, and a length of 0.74 meter. It was provided with wall ports A in three different places. E is an entrance section for the pump; D, a diffuser; the immersion heater T and the cooling coil K provide

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\*"Der Einfluss grosser Zähigkeit bei der Strömung um den Zylinder und um die Kugel", ZAMM, vol. 16, no. 3, June 1936, p. 153-164.

<sup>1</sup>The suggestion of the present work, which was prepared in the Kaiser Wilhelm Institute in Göttingen, I obtained from Herr Professor Dr. Prandtl, to whom in this place I express my most heartfelt thanks for the energetic furthering of the work and the valuable suggestions given me for its completion. Another work in the same field is published in "Forschung a.d. Gebiet des Ingenieurwesens", 1936, vol. 7, no. 1.

<sup>2</sup>M. Barker - Proc. of Royal Society, 1922, Vol. A.-101, p. 435.

temperature regulation. For impact pressure measurement a cylinder which was provided with a port was built rigidly into the test section. The diameters of the cylinders used were 1 centimeter, 1.377 centimeters, 1.953 centimeters, and 2 centimeters. The holes had a diameter of 0.1 centimeter and 0.2 centimeter. Two corrections - one because of wall effect, the other because of finite size of hole (which originated with A. Thom and first had to be checked for the measuring range under consideration here) - were applied to the measurements, which are illustrated in figure 7. The solid curve represents the theory.

More precise information on the test setup and the measurement technique are found in the work cited in footnote 1.

In the case of the measurement of static pressure on a sphere, a sphere provided with a hole was affixed to a pitot tube, the sphere having one time a diameter of 0.8 centimeter, the other time 1.6 centimeters. The execution of the measurements was in the same manner as in the case of measurement of the Barker effect. The result is shown in figure 8. The solid curve corresponds to the theory.

In order to arrive at a clearer picture of the viscous flow around a cylinder or a sphere, the case of viscous flow against a plate was next calculated. The differential equation appearing in the two-dimensional case has already been solved by Hiemenz<sup>3</sup> and will be sketched once more for the sake of a better understanding of the final form. The solution will then be used in the flow around the cylinder.

After this, the three-dimensional flow against a plate in a fluid jet will be treated, to be used on the sphere.

#### VISCOUS FLOW IN THE VICINITY OF A STAGNATION POINT

##### (TWO-DIMENSIONAL CASE)

Potential-flow theory gives for the velocity components in the neighborhood of a stagnation point, for the case of flow perpendicular to a plane wall (fig. 2):

$$U = -ax \quad V = ay$$

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<sup>3</sup>Hiemenz - Dissertation, Dingl. Polyt. Journal, V.326 (1911), No. 21-26.

The pressure is found from the Bernoulli equation to be

$$p_0 - p = \frac{\rho}{2}(U^2 + V^2) = \frac{\rho a^2}{2}(x^2 + y^2)$$

With consideration of viscosity, Hiemenz<sup>3</sup> makes the formulation:

$$u = -f(x) \quad v = yf'(x) \quad (1)$$

$$p_0 - p = \frac{\rho a^2}{2}(F(x) + y^2) \quad (2)$$

The continuity equation is fulfilled; the boundary conditions read:

for  $x = 0$  (that is, at the wall):  $u = v = 0, \quad f = f' = 0$

for  $x = \infty$ :  $v = V, \quad f' = a$

In equation (2) if  $p_0$  signifies the pressure at the stagnation point, then  $F(0) = 0$ . From the equations of motion

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

one obtains as determining equations for  $f(x)$  and  $F(x)$ :

$$ff'' = \frac{a^2}{2} F' - \nu f'''' \quad (3)$$

$$f'^2 - ff'''' = a^2 + \nu f'''' \quad (4)$$

with the boundary conditions above.

Equation (4) has already been integrated by Heimenz. In order to make the coefficients equal to unity, he sets

$$f(x) = A\phi(\xi) \quad \xi = \alpha x$$

From comparison of the coefficients, it follows that

$$A = \sqrt{va} \quad \alpha = \sqrt{\frac{a}{v}}$$

With this, equation (4) becomes

$$\phi'^2 - \phi''\phi = 1 + \phi''' \quad (5)$$

The new boundary conditions read

$$\text{for } \xi = 0: \phi = \phi' = 0$$

$$\text{for } \xi = \infty: \phi' = 1$$

The behavior of  $\phi$  and the first two derivatives is shown in figure 4.

We need the pressure difference between the stagnation point and the pressure for  $x = \infty$ . For  $x = \infty$ :

$$\phi' = 1 \quad f' = a \quad \phi = \xi - 0.647$$

From integration of equation (3),  $F$  is determined to be

$$\frac{a^2}{2} F = v f' + \frac{1}{2} f^2 \quad (6)$$

If one now forms  $(p_o - p)$  minus  $(p_o' - p')$ , as given by the Bernoulli equation, one obtains:

$$(p_o - p) - (p_o' - p') = \frac{\rho a^2}{2} (F(x) + y^2) - \frac{\rho}{2} (f_\infty^2 + y^2 f_\infty'^2)$$

For the stagnation streamline, for which  $y = 0$ :

$$(p_0 - p) - (p_0' - p') = \frac{\rho a^2}{2} F(x) \left( \frac{\rho}{2} f_\infty^2 \right)$$

If one puts in for  $F$  the previously obtained value, there results

$$(p_0 - p) - (p_0' - p') = \frac{\rho}{2} (2vf_\infty' + f_\infty^2 - f_\infty'^2) = \rho vf_\infty'$$

For  $x = \infty$ ,  $f' = a$ ; therefore, one obtains as a final formula:

$$(p_0 - p) - (p_0' - p') = \rho va \quad (6a)$$

#### VISCOUS FLOW AT A STAGNATION POINT

##### (ROTATIONALLY SYMMETRIC CASE)

For the solution of the differential equation arising, all the expressions, such as the equations of motion, the velocity components, etc., were reduced to cylindrical coordinates.

If  $z$ ,  $r$  and  $\beta$  are the coordinates (fig. 3), then corresponding to the two-dimensional case, there will apply:

$$v_z = -f'(z) \quad v_r = \frac{r}{2} f''(z) \quad (7)$$

$$p_0 - p = \frac{\rho a^2}{2} (F(z) + r^2) \quad (8)$$

The continuity equation is again fulfilled;  $v_\beta = 0$ , since we are dealing with a rotationally symmetric process. These expressions stem from the frictionless problem of a fluid jet against a plate, where

$$V_r = ar \quad V_z = -2az$$

and

$$p_0 - p = \frac{\rho a^2}{2} (4z^2 + r^2)$$

The quantity  $2az$  in the frictionless case is replaced by  $f(z)$  in the viscous case. In the case at hand the equations of motion read:

$$\left. \begin{aligned} \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) \\ \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial r^2} \right) \end{aligned} \right\} \quad (9)$$

Substituting equations (7) and (8) in equation (9) gives:

$$\frac{1}{2} f'^2 - ff'' = 2a^2 + \nu f''' \quad (10)$$

$$ff'' = \frac{a^2}{2} F' - \nu f''' \quad (11)$$

The boundary conditions read

$$\text{for } z = 0: f = f' = 0$$

$$\text{for } z = \infty: f' = 2a$$

If one finds  $f$  from equation (10), one can therewith determine  $F$  from equation (11). One next substitutes the transformation

$$f(z) = A\phi(\xi) \quad \xi = \alpha z \quad (12)$$

into equation (10), in order to make the two coefficients equal to unity. This yields:

$$\frac{1}{2} \alpha^2 A^2 \phi'^2 - \alpha^2 A^2 \phi\phi'' = 2a^2 + \nu \alpha^3 A \phi'''$$

From equating the coefficients:

$$\frac{1}{2} \alpha^2 A^2 = 2a^2 = \nu \alpha^3 A$$

$$A = 2\sqrt{av} \quad \alpha = \sqrt{\frac{a}{v}} \quad (13)$$

From equation (10) with equations (12) and equations (13) there results the final differential equation:

$$\phi'''' + 2\phi\phi'' - \phi'^2 + 1 = 0 \quad (14)$$

with the boundary conditions:

$$\text{for } \xi = 0: \phi = \phi' = 0$$

$$\text{for } \xi = \infty: \phi' = 1$$

The differential equation (14), just as Hiemenz', is no longer elementarily integrable. Its solution was obtained, accordingly, through a power series development from zero:

$$\phi = a_0 + a_1\xi + a_2\xi^2 + \dots + a_n\xi^n \quad (15)$$

By the method of undetermined coefficients,  $a_1$  can be determined. Since, however, one boundary condition lies at infinity, one coefficient remains undetermined; and in fact it turns out to be  $a_2$ . From the recursion formulas

$$\phi = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \dots$$

$$\phi' = a_1 + 2a_2\xi + 3a_3\xi^2 + \dots$$

$$\phi'' = 2a_2 + 2 \times 3a_3\xi + 3 \times 4a_4\xi^2 + \dots$$

result as coefficients:



$$a_0 = a_1 = 0$$

$$a_2 = \text{for the present undetermined}$$

$$a_3 = -0.166667$$

$$a_4 = 0$$

$$a_5 = 0$$

$$a_6 = 0.555556 \times 10^{-2} a_2$$

$$a_7 = -0.396825 \times 10^{-3}$$

$$a_8 = 0$$

$$a_9 = -0.440917 \times 10^{-3} a_2^2$$

$$a_{10} = 0.793651 \times 10^{-4} a_2$$

$$a_{11} = -0.360750 \times 10^{-5}$$

$$a_{12} = 0.374111 \times 10^{-4} a_2^3$$

$$a_{13} = -0.114597 \times 10^{-4} a_2^2$$

$$a_{14} = 0.115735 \times 10^{-5} a_2$$

$$a_{15} = -0.301482 \times 10^{-5} a_2^4 - 0.385784 \times 10^{-7}$$

$$a_{16} = 0.134896 \times 10^{-5} a_2^3$$

$$a_{17} = -0.211005 \times 10^{-6} a_2^2$$

$$a_{18} = 0.224141 \times 10^{-6} a_2^5 + 0.157758 \times 10^{-7} a_2$$

$$a_{19} = -0.135546 \times 10^{-6} a_2^4 - 0.415153 \times 10^{-9}$$

$$a_{20} = 0.316633 \times 10^{-7} a_2^3$$

$$a_{21} = -0.152798 \times 10^{-7} a_2^6 - 0.295658 \times 10^{-8} a_2^2$$

$$a_{22} = 0.119505 \times 10^{-7} a_2^5 + 0.199390 \times 10^{-9} a_2$$

$$a_{23} = -0.371665 \times 10^{-8} a_2^4 - 0.433457 \times 10^{-11}$$

$$a_{24} = 0.956242 \times 10^{-9} a_2^7 + 0.554360 \times 10^{-9} a_2^3$$

$$a_{25} = -0.943031 \times 10^{-9} a_2^6 - 0.462914 \times 10^{-11} a_2^2$$

In order now to be able to determine  $a_2$ , a second series development from infinity was set up, which was adjusted to the boundary condition for  $\phi$  at infinity. To this end one sets

$$\phi = \phi_0 + \phi_1 \quad (16)$$

in which  $\phi_1$  corresponds to a small quantity, which one can neglect in the following expressions when it appears squared.  $\phi_0$  is the solution for  $\xi = \infty$ .

$$\phi' = \phi_0' + \phi_1' = 1 + \phi_1'$$

since for  $\xi = \infty$ :

$$\phi_0' = \phi' = 1 \quad \phi_1'' = \phi_1'' \quad \phi_1''' = \phi_1''' \quad (16a)$$

The boundary condition reads

$$\text{for } \xi = \infty: \phi_1' = 0$$

Furthermore,  $\phi_0 = \xi$ .

The integration constant is omitted, since in the following calculation it comes in again automatically.

If one substitutes the above values into equation (10), one obtains

$$\phi_1''' + 2(\phi_0\phi_1'' + \phi_1\phi_1'') - (\phi_0'^2 + 2\phi_0'\phi_1' + \phi_1'^2) + 1 = 0 \quad (17)$$

Or if one neglects the squared terms in  $\phi_1$ :

$$\phi_1''' + 2\xi\phi_1'' - 2\phi_1' = 0 \quad (18)$$

To solve this differential equation one sets

$$\Phi = \phi_1'$$

With this, equation (18) gives

$$\Phi'' + 2\xi\Phi' - 2\Phi = 0 \quad (19)$$

A special, not identically vanishing solution of equation (19) is:

$$\phi_1 = \xi$$

If  $\phi_2$  is an additional solution, then

$$\phi_1\phi_2' - \phi_1'\phi_2 = e^{-\int_{\infty}^{\xi} 2\xi d\xi}$$

$$\xi\phi_2' - \phi_2 = e^{-\xi^2}$$

$$\phi_2' - \frac{1}{\xi}\phi_2 - \frac{1}{\xi}e^{-\xi^2} = 0$$

This equation is directly solvable. Its solution is

$$\phi_2 = \xi \int_{\infty}^{\xi} \frac{1}{\eta^2} e^{-\eta^2} d\eta$$

The general solution of equation (19) is then:

$$\Phi = C_1\phi_1 + C_2\phi_2 = C_1\xi + C_2\xi \int_{\infty}^{\xi} \frac{1}{\eta^2} e^{-\eta^2} d\eta$$

Since for  $\xi = \infty$ ,  $\phi_1' = \phi = 0$ , then  $C_1 = 0$ . Therefore

$$\phi_1' = \phi = C_2 \xi \int_{\infty}^{\xi} \frac{1}{\eta^2} e^{-\eta^2} d\eta = C_2 \xi \left[ -\frac{1}{\xi} e^{-\xi^2} - 2 \int_{\infty}^{\xi} e^{-\eta^2} d\eta \right]$$

and since  $\phi_1 = \int_{\infty}^{\xi} \phi d\xi$

$$\begin{aligned} \phi_1 &= C_3 + C_2 \int_{\infty}^{\xi} \left[ -e^{-\eta^2} - 2\eta \int_{\infty}^{\eta} e^{-u^2} du \right] d\eta \\ &= C_3 - C_2 \int_{\infty}^{\xi} e^{-\eta^2} d\eta - 2C_2 \int_{\infty}^{\xi} \eta d\eta \int_{\infty}^{\eta} e^{-u^2} du \end{aligned} \quad (20)$$

The double integral becomes, according to Blasius<sup>4</sup>,

$$2C_2 \int_{\infty}^{\xi} \eta d\eta \int_{\infty}^{\eta} e^{-u^2} du = 2C_2 \left[ \frac{1}{2} \xi^2 \int_{\infty}^{\xi} e^{-\eta^2} d\eta - \frac{1}{4} \int_{\infty}^{\xi} e^{-\eta^2} d\eta + \frac{1}{4} \xi e^{-\xi^2} \right]$$

With this, equation (20) becomes:

$$\frac{\phi_1 - C_3}{C_2} = -\frac{1}{2} \xi e^{-\xi^2} - \left( \frac{1}{2} + \xi^2 \right) \int_{\infty}^{\xi} e^{-\eta^2} d\eta \quad (21)$$

Now:

$$\left. \begin{aligned} 2 \int_{\infty}^{\xi} e^{-\eta^2} d\eta &= 2 \int_0^{\xi} e^{-\eta^2} d\eta - 2 \int_0^{\infty} e^{-\eta^2} d\eta = 2 \int_0^{\xi} e^{-\eta^2} d\eta - \sqrt{\pi} \\ \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\eta^2} d\eta &= \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\eta^2} d\eta - 1 \end{aligned} \right\} \quad (22)$$

<sup>4</sup>Blasius - Dissertation, Z.F.M.u. Phys. 1908, p. 1.

If one substitutes equation (22) in equation (21), one can calculate  $\phi$  pointwise, since

$$\frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\eta^2} d\eta$$

is tabulated. Therefore

$$\frac{\phi_1 - c_3}{c_2} = -\frac{1}{2} \xi e^{-\xi^2} - \left(\frac{1}{2} + \xi^2\right) \frac{\sqrt{\pi}}{2} \left\{ \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\eta^2} d\eta - 1 \right\}$$

$$\frac{\phi_1'}{c_2} = -e^{-\xi^2} - \xi \sqrt{\pi} \left\{ \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\eta^2} d\eta - 1 \right\}$$

$$\frac{\phi_1''}{c_2} = \sqrt{\pi} \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-\eta^2} d\eta \right\}$$

Herewith  $\phi$ ,  $\phi'$ , and  $\phi''$  are determined for the development at infinity, as a comparison with equations (16) and equation (16a) shows.

In both developments  $a_2$ ,  $c_2$ , and  $c_3$  appear as unknowns. If one now combines both solutions at the point  $\xi = \xi_0$ , and determines that the value of the function and the first two derivatives of the series development at zero are equal to the corresponding values that one obtains from the development at infinity, then three determining

equations for  $a_2$ ,  $C_2$  and  $C_3$  result, from which the unknowns can be determined. Therefore:

$$a_0 + a_1 \xi_0 + \dots + a_{25} \xi_0^{25} = C_2 \left\{ -\frac{1}{2} \xi_0 e^{-\xi_0^2} - \left( \frac{1}{2} + \xi_0^2 \right) \frac{\sqrt{\pi}}{2} \left[ \frac{2}{\sqrt{\pi}} \int_0^{\xi_0} e^{-\eta^2} d\eta - 1 \right] \right\} + C_3 + \xi_0$$

$$a_1 + 2a_2 \xi_0 + \dots + 25a_{25} \xi_0^{24} = C_2 \left\{ -e^{-\xi_0^2} - \xi_0 \sqrt{\pi} \left[ \frac{2}{\sqrt{\pi}} \int_0^{\xi_0} e^{-\eta^2} d\eta - 1 \right] \right\} + 1$$

$$2a_2 + 6a_3 \xi_0 + \dots + 24 \times 25a_{25} \xi_0^{23} = C_2 \sqrt{\pi} \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^{\xi_0} e^{-\eta^2} d\eta \right\}$$

$\xi_0$  was chosen 1.8, since  $\phi''$ , from which  $\phi$  is built up, can then be determined accurately to 0.002, on account of the alternating signs of the power series development.

The solution of the determining equations gives

$$a_2 = 0.658619, \quad C_2 = 2.16492, \quad C_3 = -0.557611$$

Thereby  $a_2$  is determined accurately to at least five places. For the coefficients of the power series this yields:

$$\begin{aligned}
 a_2 &= 0.658619 & a_{16} &= 0.385391 \times 10^{-6} \\
 a_3 &= -0.166667 & a_{17} &= -0.958673 \times 10^{-7} \\
 a_6 &= 0.365900 \times 10^{-2} & a_{18} &= 0.381678 \times 10^{-7} \\
 a_7 &= -0.396825 \times 10^{-3} & a_{19} &= -0.259200 \times 10^{-7} \\
 a_9 &= -0.191261 \times 10^{-3} & a_{20} &= 0.904605 \times 10^{-8} \\
 a_{10} &= 0.522714 \times 10^{-4} & a_{21} &= -0.252966 \times 10^{-8} \\
 a_{11} &= -0.360750 \times 10^{-5} & a_{22} &= 0.161233 \times 10^{-8} \\
 a_{12} &= 0.106882 \times 10^{-4} & a_{23} &= -0.703675 \times 10^{-9} \\
 a_{13} &= -0.497098 \times 10^{-5} & a_{24} &= 0.209783 \times 10^{-9} \\
 a_{14} &= 0.762253 \times 10^{-6} & a_{25} &= -0.970520 \times 10^{-10} \\
 a_{15} &= -0.605859 \times 10^{-6}
 \end{aligned}$$

$$a_0 = a_1 = a_4 = a_5 = a_8 = 0$$

The values for  $\phi$ ,  $\phi'$ , and  $\phi''$  are shown more accurately however, in Table I. In this case,  $\phi''$  is calculated accurately to two decimal places,  $\phi'$  to two, and  $\phi$  to three. With this the differential equation (14) is solved.

From integration of equation (11), one obtains

$$\frac{a^2}{2} F = v f' + \frac{1}{2} f^2 = 2av(\phi' + \phi^2) \quad (23)$$

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As in the plane case, one uses again the pressure difference between the stagnation pressure and the pressure for  $z = \infty$ . For  $\xi = 0$ , equation (23) is equal to zero; for  $\xi = \infty$

$$\phi' = 1 \quad f' = 2a \quad \phi = \xi - 0.557611$$

If one now forms again  $(p_o - p)$  minus  $(p_o' - p')$ , as given by the Bernoulli equation, one obtains:

$$(p_o - p) - (p_o' - p') = \rho \left( v f_{\infty}' + \frac{1}{2} f_{\infty}^2 - \frac{1}{2} f_{\infty}'^2 \right) = \rho v f_{\infty}'$$

As a final formula one obtains

$$(p_o - p) - (p_o' - p') = 2\rho va \quad (24)$$

#### STAGNATION PRESSURE ON A CYLINDER

For the stagnation streamline the Navier-Stokes differential equation gives

$$u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2} \quad (25)$$

Figure 12 shows the variation of  $u$  on this streamline. The different behavior of  $u$  at the stagnation point from potential flow is explained by the influence of viscosity. If one integrates between the boundary  $R$  and  $\infty$ , one gets

$$\frac{u_R^2}{2} - \frac{u_{\infty}^2}{2} + \frac{1}{\rho}(p_R - p_{\infty}) = \nu \left. \frac{\partial u}{\partial x} \right|_{\infty}^R + \nu \int_{\infty}^R \frac{\partial^2 u}{\partial y^2} dx$$

Figure 13 shows that

$$\nu \left. \frac{\partial u}{\partial x} \right|_{\infty}^R = 0$$



likewise  $u_R = 0$ , and we want to identify  $p_R$  with  $p_o$  of the preceding calculation. One obtains, therefore

$$(p_o - p) - \frac{\rho U_o^2}{2} = \rho v \int_{\infty}^R \frac{\partial^2 u}{\partial y^2} dx \quad (26)$$

$$\int_{\infty}^R \frac{\partial^2 u}{\partial y^2} dx$$

is calculated approximately in that for  $u$  the value corresponding to the potential flow is put in. The contribution of the boundary layer to the integral is, in the case of not too small Reynolds number, small in comparison. As potential function  $\Phi$  of the flow around the cylinder, one obtains

$$\Phi = U_o \left( x + \frac{xR^2}{r^3} \right)$$

and with it:

$$\frac{\partial^2 u}{\partial y^2} = -U_o \left( -\frac{2R^2}{r^4} + \frac{8x^2R^2}{r^6} + \frac{8y^2R^2}{r^6} - \frac{48x^2y^2R^2}{r^8} \right)$$

For  $y = 0$ , therefore, along the stagnation streamline:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{6U_oR^2}{x^4}$$

$$\rho v \int_{\infty}^R \frac{\partial^2 u}{\partial y^2} dx = \frac{2\rho v U_o}{R}$$

Herewith equation (26) gives

$$(p_o - p) = \frac{\rho U_o^2}{2} + \frac{2\rho v U_o}{R} \quad (27a)$$

or

$$\frac{p_o - p}{\rho U_o^2/2} = 1 + \frac{4}{Re} \quad (27)$$

If one substitutes  $\gamma = \rho g$ , then formula (27) reads

$$\frac{p_o - p}{\gamma U_o^2/2g} = 1 + \frac{4}{Re} \quad (27b)$$

where  $g = 9.81$  meters per second<sup>2</sup>.

In order to be able to accomplish a comparison of test results with theory, the "displacement thickness" (see Tolmien: Hdb. d Experimentalphysik, v. 4, 1st part, p. 262, "Grenzschichttheorie (Boundary Layer Theory)") on the cylinder must yet be considered in the calculation. Solution of the differential equation (5) yields (fig. 5):

$$\xi^* = \alpha x^* = \alpha \delta^* = 0.647$$

where  $\delta^*$  is the displacement thickness. Therefore

$$0.647 = \delta^* \sqrt{\frac{a}{v}} \quad (28)$$

If one compares the flow in the region nearest the stagnation point for the cylinder and for the flow against a plate, one obtains from equations (27a) and (6a)

$$\rho v a = \frac{2\rho v U_o}{R}, \quad a = \frac{2U_o}{R}$$

If one substitutes this value in equation (28), one obtains for the displacement thickness

$$0.647 = \sqrt{\frac{2U_o}{Rv}} \delta^* = \frac{\delta^*}{R} \sqrt{2Re}$$

$$\frac{\delta^*}{R} = \frac{0.457}{\sqrt{Re}}$$

The dependence of  $\delta^*/R$  on  $Re$  is shown in figure 6.

Now since in the test results  $Re$  is formed from the cylinder radius  $R$ , the actual effective radius is therefore  $(R + \delta^*)$ , and equation (27b) is altered to:

$$\frac{p_o - p}{\gamma U_o^2 / 2g} = 1 + \frac{4\nu}{U_o(R + \delta^*)}$$

With this one obtains as a final rule for the stagnation pressure on the cylinder

$$\frac{p_o - p}{\gamma U_o^2 / 2g} = 1 + \frac{4}{Re + 0.457\sqrt{Re}} \quad (29)$$

In figure 7 the solid curve again gives the theory, which agrees very well with the practice.

#### STAGNATION PRESSURE ON A SPHERE

Corresponding to a cylinder, for the stagnation streamline of a sphere

$$u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + \nu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

If one integrates again over  $x$  from  $\infty$  to  $R$ :

$$(p_o - p) = \frac{\rho U_o^2}{2} + \rho \nu \int_{\infty}^R \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dx \quad (30)$$

since

$$\int_{\infty}^R \frac{\partial^2 u}{\partial x^2} dx = 0, \quad U_R = 0$$

The integral on the right side of equation (30) one again solves by

substituting for  $u$  the value for the potential flow. The potential function of this flow is

$$\Phi = -U_o \left( x + \frac{xR^3}{2r^3} \right)$$

For potential flow it is further true that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Therefore

$$\int_{-\infty}^R \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dx = \int_R^{\infty} \frac{\partial^2 u}{\partial x^2} dx = \frac{\partial u}{\partial x} \Big|_R^{\infty}$$

If one substitutes in the last formula the value for  $\partial u / \partial x$  given by  $\Phi$ , one obtains

$$\int_{-\infty}^R \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dx = \frac{3U_o}{R}$$

Herewith equation (30) becomes

$$p_o - p = \frac{\rho U_o^2}{2} + \frac{3\rho v U_o}{R} \quad (31a)$$

or

$$\frac{p_o - p}{\rho U_o^2 / 2} = 1 + \frac{6}{Re} \quad (31)$$

or with  $\gamma = \rho g$

$$\frac{p_o - p}{\gamma U_o^2 / 2g} = 1 + \frac{6}{Re} \quad (31b)$$

If here one also puts the displacement thickness into the calculation, one gets (since in the rotationally symmetric case  $\xi^* = 0.5576$ ):

$$0.5576 = \delta^* \sqrt{a/\nu}$$

Comparison of equation (24) and equation (31a) yields

$$\left. \begin{aligned} u &= \frac{3U_0}{2R} & 0.5576 &= \sqrt{\frac{3U_0}{2\nu R}} \delta^* \\ \frac{\delta^*}{R} &= \frac{0.455}{\sqrt{Re}} \end{aligned} \right\} \quad (32)$$

It appears that the displacement thickness for a sphere and a cylinder are equal within  $\frac{1}{2}$  percent, although the displacement thickness in the case of plane flow against a plate is different from the corresponding three-dimensional flow.

If one considers the displacement thickness in equation (31b), one obtains as a final stagnation pressure formula for a sphere

$$\frac{p_0 - p}{\gamma U_0^2 / 2g} = 1 + \frac{6}{Re + 0.455\sqrt{Re}} \quad (33)$$

The solid curve in figure 8 corresponds to the theory; the agreement with test results is again satisfactory.

From the final stagnation pressure formula the dependence of the numerical factor  $\epsilon$  on  $Re$  can be determined, if one sets

$$\frac{p_0 - p}{\gamma U_0^2 / 2g} = 1 + \frac{\epsilon}{Re}$$

For the sphere there results

$$\epsilon = \frac{6 Re}{Re + 0.455\sqrt{Re}} \quad (34)$$

From Stokes' calculation one obtains for small  $Re$ :  $\epsilon = 3$ . In figure 9 is drawn  $\log Re$  as abscissa,  $\epsilon$  as ordinate. In the region from about  $Re = 0.1$  to  $Re = 1$  the course of  $\epsilon$  is essentially different, since Stokes' law describes an approximation for very small Reynolds number and the above law is an approximation for large Reynolds number.

For the cylinder one obtains in the same fashion

$$\epsilon = \frac{4Re}{Re + 0.457\sqrt{Re}} \quad (35)$$

According to Lamb<sup>5</sup>, for small  $Re$ , for which the validity of the formula extends to about  $Re = 0.5$ :

$$\epsilon = \frac{Re + 4}{1.309 - 7\ln Re} - Re \quad (36)$$

In figure 10 is again shown the dependence of  $\epsilon$  on  $\log Re$ . Within the accuracy of measurement the test results here also confirm equation (29).

With the help of the flow against a plate it is now also possible to establish approximately the course of  $u$ ,  $\partial u / \partial x$ , and from this  $p$ , on the stagnation streamline. A single curve was assumed in which, inside the displacement thickness  $\delta^*$ , the magnitudes as given by the flow against a plate were used. From the displacement thickness on, which had a value of 0.0455 in the foregoing case for  $Re = 100$ , the potential flow was calculated. To explain the transition from viscous to potential flow, I would like to go through the calculation of  $u$  as an example. The solution of the viscous problem  $u_1/u_0$  has as asymptote the tangent to the curve  $u_2/u_0$ , which was determined from potential theory, at the point  $\delta^* = 0.0455$  centimeter. In figure 11 this tangent is labelled  $t$ . The difference  $k$  between the asymptote  $t$  and  $u_1$  at the point  $x_0$  gives the deviation of viscous flow from potential flow at this point. Therefore to the value  $u_1$  at the point  $x_0$  was added the proper  $k$ . With the help of this procedure one obtains point-wise the transition from  $u_1$  to  $u_2$ .

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<sup>5</sup>Lamb - "Hydrodynamics" (2nd Edition 1931; German Edition by E. Helly, p. 696, par. 343).

$\partial u / \partial x$  was determined correspondingly; the pressure  $p$  was found from the equations of motion to be, in the case of the sphere:

$$\frac{p_0 - p}{\gamma U_0^2 / 2g} = 1 - \frac{U_x^2}{400} - \frac{1}{200} \frac{\partial u_x}{\partial x} + \frac{6}{U_0(R + x)^4} \quad (37)$$

Instead of 6 in the last term of the preceding equation (37), in the case of the cylinder one gets the factor 4. Figures 12, 13, and 14 are the results; by way of comparison the corresponding curves for the cylinder and the sphere are shown on one sheet. The curves are true, as already said, for  $Re = 100$ , in which  $R = 0.01$  meter;  $U_0 = 1$  meter;  $\nu = 0.0001$  kilogram  $\times$  second per meter<sup>2</sup> was assumed.

Naturally the last curves give only an approximation, which can be made essentially better through a second approximation; yet this task in the framework of the foregoing work would lead too far.

#### SUMMARY

In the foregoing work the stagnation pressure increase on cylinders and spheres brought about through the influence of large viscosity, was reported on.

For the three-dimensional problem, hence the flow around a sphere, a differential equation was set up which corresponded to that of Heimenz, who had already solved the two-dimensional case. The solution was ascertained likewise through an approximate method. The solutions for the two- and for the three-dimensional case were used for the flow around the cylinder and sphere respectively; the formulas so obtained for the stagnation pressure increase stood in good agreement with the reported test results. Finally, a procedure to determine the velocity and pressure variation, as well as the variation of  $\partial u / \partial x$  on the stagnation streamline was shown and used on the practical case of  $Re = 100$ .

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TABLE I

$\xi$	$\phi$	$\phi'$	$\phi''$	$\xi$	$\phi$	$\phi'$	$\phi''$
0	0	0	1.3172	1.4	0.8546	0.9476	0.1697
.1	.0064	.1267	1.2172	1.5	.9502	.9635	.1301
.2	.0250	.2434	1.1173	1.6	1.0472	.9762	.0895
.3	.0548	.3502	1.0181	1.7	1.1453	.9863	.0622
.4	.0974	.4471	.9200	1.8	1.2424	.9905	.0418
.5	.1439	.5343	.8235	1.9	1.3436	.9935	.0276
.6	.2012	.6129	.7298	2.0	1.4430	.9962	.0180
.7	.2659	.6804	.6400	2.1	1.5413	.9979	.0115
.8	.3370	.7400	.5548	2.2	1.6409	.9986	.0073
.9	.4137	.7847	.4742	2.3	1.7416	.9991	.0042
1.0	.4951	.8352	.4015	2.4	1.8417	.9995	.0027
1.1	.5805	.8712	.3351	2.5	1.9420	.9997	.0016
1.2	.6653	.9025	.2760	2.6	2.0423	.9999	.0009
1.3	.7608	.9247	.2241				



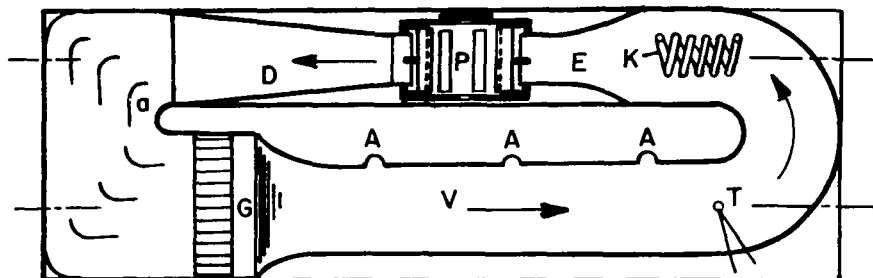


Figure 1.- Test tunnel.

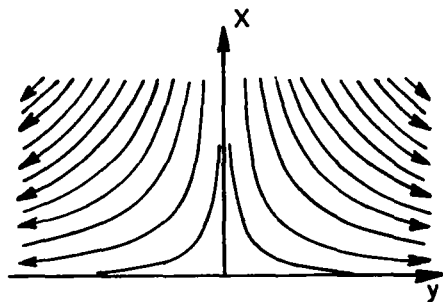


Figure 2.- Streamline picture for flow against a plate (two-dimensional).

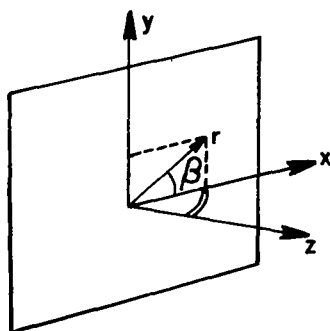


Figure 3.

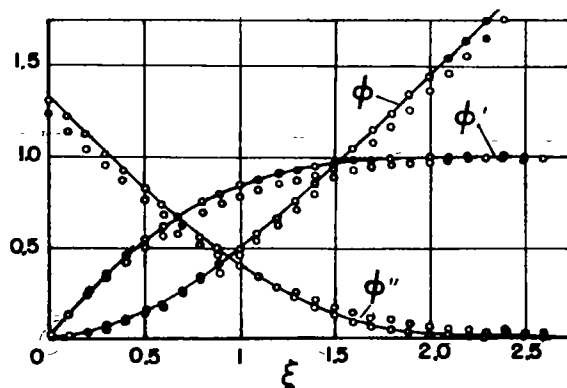


Figure 4.-  $\phi, \phi', \phi''$ . The curves drawn out illustrate the two-dimensional solution, those not drawn out the three-dimensional.

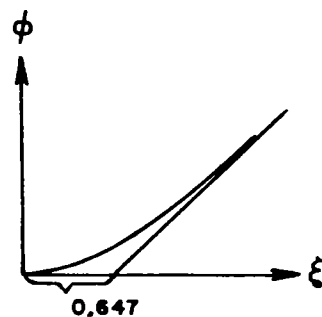


Figure 5.

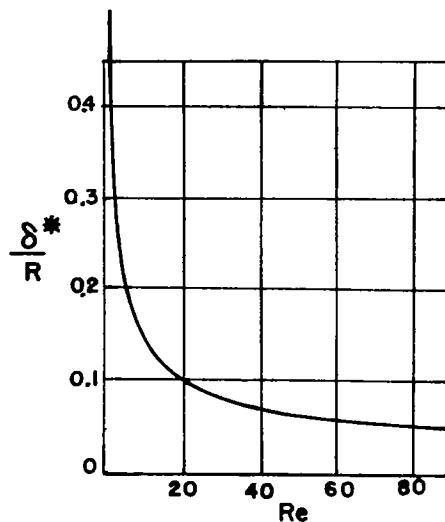


Figure 6.- Momentum thickness on a cylinder  $\frac{\delta^*}{R} = \frac{0.457}{\sqrt{Re}}$ ; and on a sphere  $\frac{\delta^*}{R} = \frac{0.455}{\sqrt{Re}}$ .

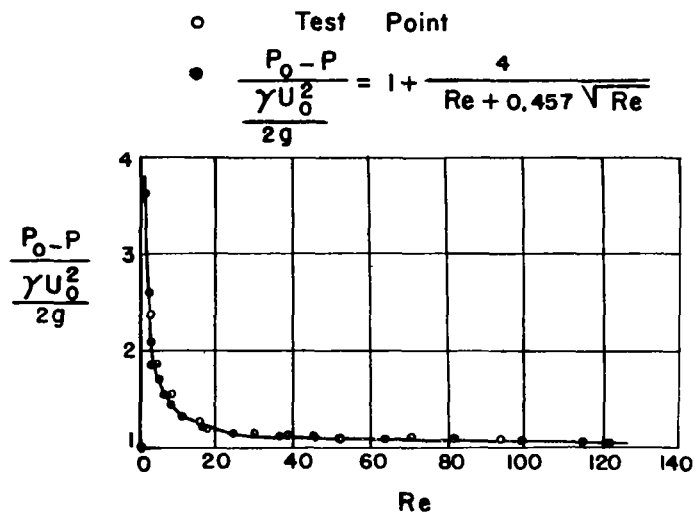


Figure 7.- Stagnation pressure on cylinder.

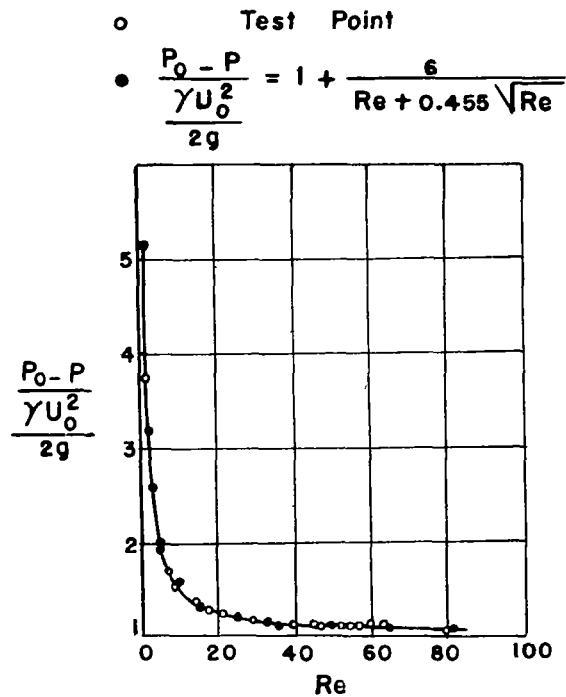


Figure 8.- Stagnation pressure on sphere.

o Test Point

Curve I:  $\epsilon = 3$  (stokes' solution)

Curve II:  $\epsilon = \frac{6Re}{Re + 0.455\sqrt{Re}}$

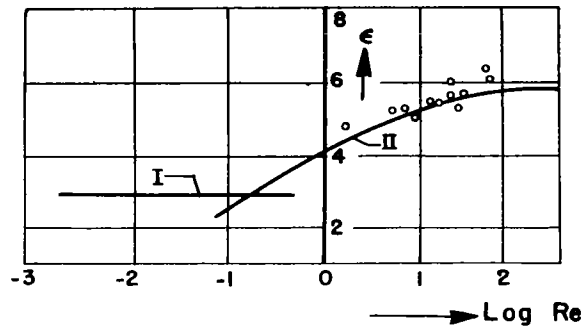


Figure 9.-  $\epsilon$  for sphere.

o Test Point

Curve I:  $\epsilon = \frac{Re + 4}{1.309 \ln Re} Re$  (Oseens' solution)

Curve II:  $\epsilon = \frac{4Re}{Re + 0.457\sqrt{Re}}$

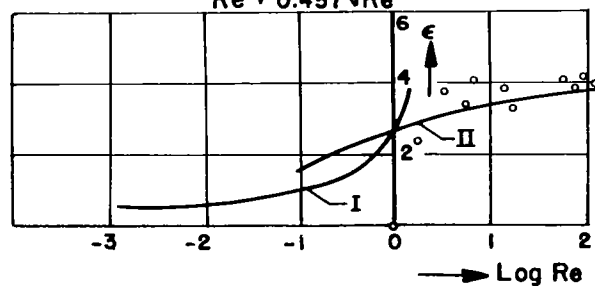


Figure 10.-  $\epsilon$  for cylinder.

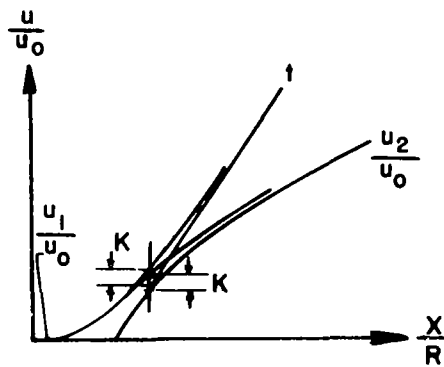


Figure 11.- Illustration of interpolation for transition from viscous to potential flow.

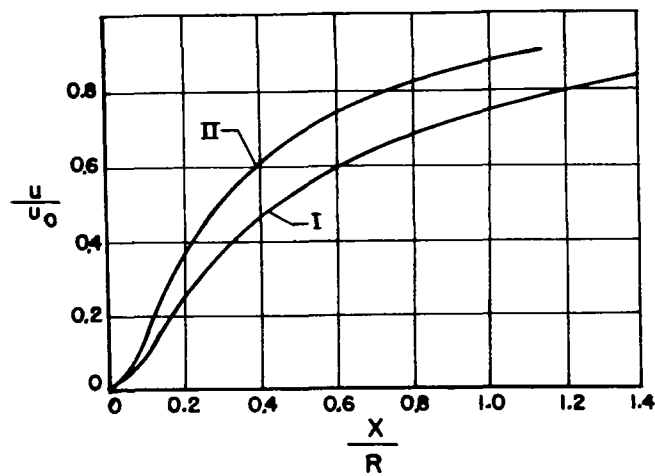


Figure 12.- Velocity variation on stagnation streamline; cylinder: Curve I; sphere: Curve II.  $Re = 100$ .

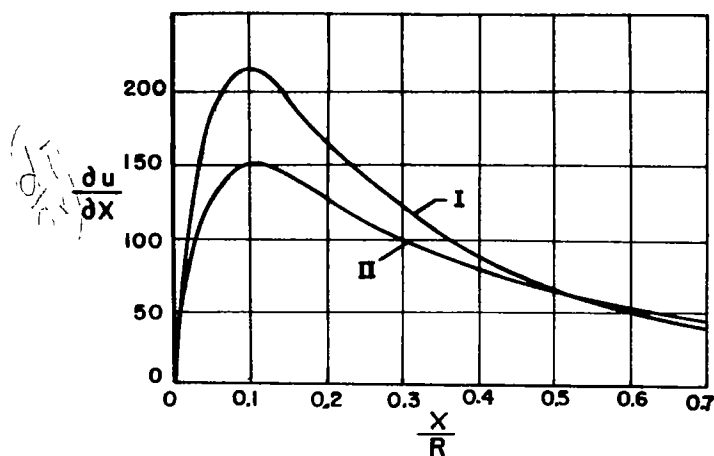


Figure 13.- Variation of  $\frac{\partial u}{\partial x}$  on stagnation streamline; cylinder: Curve I; sphere: Curve II.  $Re = 100$ .

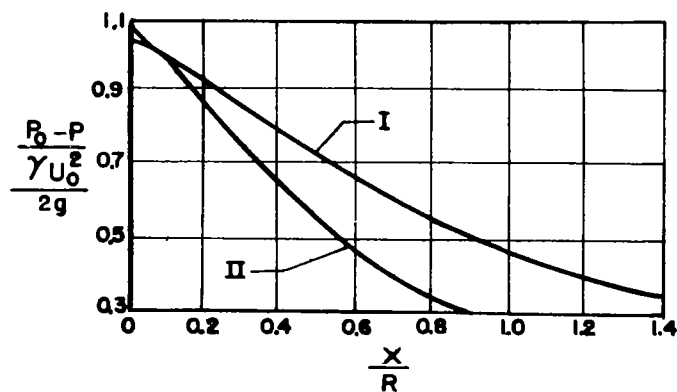


Figure 14.- Pressure variation on stagnation streamline; cylinder: Curve I; sphere: Curve II.  $Re = 100$ .

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